

Infinite words rich and almost rich in generalized palindromes

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Abstract. We focus on Θ -rich and almost Θ -rich words over a finite alphabet \mathcal{A} , where Θ is an involutive antimorphism over \mathcal{A}^* . We show that any recurrent almost Θ -rich word \mathbf{u} is an image of a recurrent Θ' -rich word under a suitable morphism, where Θ' is again an involutive antimorphism. Moreover, if the word \mathbf{u} is uniformly recurrent, we show that Θ' can be set to the reversal mapping. We also treat one special case of almost Θ -rich words. We show that every Θ -standard words with seed is an image of an Arnoux-Rauzy word.

Keywords: palindrome, palindromic defect, richness

1 Introduction

In this paper we will deal with infinite words over a finite alphabet \mathcal{A} . A word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ we are interested in has its language $\mathcal{L}(\mathbf{u})$ saturated, in a certain sense, by generalized palindromes, here called Θ -palindromes. We will use the symbol Θ for an involutive antimorphism, i.e., a mapping $\Theta : \mathcal{A}^* \mapsto \mathcal{A}^*$ such that $\Theta^2 = \text{Id}$ and $\Theta(uv) = \Theta(v)\Theta(u)$ for all $u, v \in \mathcal{A}^*$. Fixed points of Θ are called Θ -palindromes. A word w is a Θ -palindrome if $\Theta(w) = w$. The most common antimorphism used in combinatorics on words is the reversal mapping. We will denote it by Θ_0 . The reversal mapping associates to every word $w = w_1w_2 \dots w_n$ its mirror image $\Theta_0(w) = w_nw_{n-1} \dots w_1$. In the case $w = \Theta_0(w)$, we will sometimes say that w is a palindrome or classical palindrome instead of Θ_0 -palindrome.

The set of distinct Θ -palindromes occurring in a finite word w will be denoted $\text{Pal}_{\Theta}(w)$. Since the empty word ε is a Θ -palindrome for any Θ , we have a simple lower bound $\#\text{Pal}_{\Theta}(w) \geq 1$.

In 2001, Droubay et al. gave in [11] an upper bound for the reversal mapping Θ_0 . They deduced that $\#\text{Pal}_{\Theta_0}(w) \leq |w| + 1$, where $|w|$ denotes the length of the word w . In [4], Brlek et al. studied involutive antimorphisms with no fixed points of length 1. For such Θ they diminished the upper bound, they showed that $\#\text{Pal}_{\Theta}(w) \leq |w|$ for all non-empty word w . In [13], the upper bound is precised. The following estimate is valid for any involutive antimorphism Θ :

$$\#\text{Pal}_{\Theta}(w) \leq |w| + 1 - \gamma_{\Theta}(w), \quad (1)$$

where $\gamma_\Theta(w) := \#\{\{a, \Theta(a)\} \mid a \text{ occurs in } w \text{ and } a \neq \Theta(a)\}$. Let us note that if $\Theta = \Theta_0$, then $\gamma_\Theta(w) = 0$, and the upper bound in (1) is the same bound as for usual palindromes.

According to the terminology for classical palindromes introduced in [12] and for Θ -palindromes in [13], we will say that a finite word w is Θ -rich if the equality in (1) holds. An infinite word $\mathbf{u} \in \mathcal{A}^\mathbb{N}$ is Θ -rich if any its factor $w \in \mathcal{L}(\mathbf{u})$ is Θ -rich. In [5], the authors introduced the *palindromic defect* of a finite word w as the difference between the upper bound $|w| + 1$ and the actual number of palindromic factors. We define analogously the Θ -palindromic defect of w as

$$D_\Theta(w) := |w| + 1 - \gamma_\Theta(w) - \#\text{Pal}_\Theta(w).$$

We define for an infinite word \mathbf{u} its Θ -palindromic defect as

$$D_\Theta(\mathbf{u}) = \sup\{D_\Theta(w) \mid w \in \mathcal{L}(\mathbf{u})\}.$$

Words with finite Θ -palindromic defect will be referred to as *almost Θ -rich*.

In [10], it is shown that rich words (i.e. Θ_0 -rich words) can be characterized using an inequality shown in [2] for infinite words with languages closed under reversal. Results of both mentioned papers were generalized in [13] for an arbitrary involutive antimorphism. In particular, it is shown that if an infinite word has its language closed under Θ , the following inequality holds

$$\mathcal{C}(n+1) - \mathcal{C}(n) + 2 \geq \mathcal{P}_\Theta(n) + \mathcal{P}_\Theta(n+1) \text{ for all } n \geq 1, \quad (2)$$

where $\mathcal{C}(n)$ is the *factor complexity* defined by $\mathcal{C}(n) := \#\{w \in \mathcal{L}(\mathbf{u}) \mid n = |w|\}$ and $\mathcal{P}_\Theta(n)$ is the Θ -palindromic complexity defined by $\mathcal{P}_\Theta(n) := \#\{w \in \mathcal{L}(\mathbf{u}) \mid w = \Theta(w) \text{ and } n = |w|\}$. The gap between the left-hand side and the right-hand side in (2) decides about Θ -richness. Let us therefore denote by $T_\Theta(n)$ the quantity

$$T_\Theta(n) := \mathcal{C}(n+1) - \mathcal{C}(n) + 2 - \mathcal{P}_\Theta(n+1) - \mathcal{P}_\Theta(n).$$

In [13], it is also shown that an infinite word with language closed under Θ is Θ -rich if and only if

$$T_\Theta(n) = 0 \text{ for all } n \geq 1.$$

The list of infinite words which are Θ_0 -rich is quite extensive. See for instance [2,7,9]. Examples of Θ -rich words can be found in [1]. Fewer examples of words with finite non-zero palindromic defect are known. Periodic words with finite non-zero Θ_0 -defect can be found in [5], aperiodic ones are studied in [12] and [3]. To our knowledge, examples of words with $0 < D_\Theta(\mathbf{u}) < +\infty$ and $\Theta \neq \Theta_0$ have not yet been explicitly exhibited. As we will show, such examples are Θ -standard words with seed defined in [8] and thus also their subset, standard Θ -episturmian words, which can be constructed from standard episturmian words (see [6]).

The main aim of this paper is to show that among words with finite Θ -palindromic defect, Θ -rich words, i.e. words with $D_\Theta(\mathbf{u}) = 0$, play an important role. We will show the following theorems.

Theorem 1. *Let $\Theta_1 : \mathcal{A}^* \mapsto \mathcal{A}^*$ be an involutive antimorphism. Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be an infinite recurrent word such that $D_{\Theta_1}(\mathbf{u}) < +\infty$. Then there exist an involutive antimorphism $\Theta_2 : \mathcal{B}^* \mapsto \mathcal{B}^*$, a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$ and an infinite recurrent word $\mathbf{v} \in \mathcal{B}^{\mathbb{N}}$ such that*

$$\mathbf{u} = \varphi(\mathbf{v}) \text{ and } \mathbf{v} \text{ is } \Theta_2\text{-rich.}$$

A stronger statement can be shown if the requirement of uniform recurrence is imposed on the word \mathbf{u} .

Theorem 2. *Let $\Theta : \mathcal{A}^* \mapsto \mathcal{A}^*$ be an involutive antimorphism. Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be an infinite uniformly recurrent word such that $D_{\Theta}(\mathbf{u}) < +\infty$. Then there exist a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$ and an infinite uniformly recurrent word $\mathbf{v} \in \mathcal{B}^{\mathbb{N}}$ such that*

$$\mathbf{u} = \varphi(\mathbf{v}) \text{ and } \mathbf{v} \text{ is } \Theta_0\text{-rich.}$$

One can conclude that rich words, using the classical notion of palindrome, play somewhat more important role than Θ -rich words for an arbitrary $\Theta \neq \Theta_0$.

Proofs of the two stated theorems do not provide any relation between the size of the alphabet \mathcal{B} of the word \mathbf{v} and the size of the original alphabet \mathcal{A} . In the following special case, the size of \mathcal{B} can be bounded. Moreover, the word \mathbf{v} is more specific.

Theorem 3. *Let $\Theta : \mathcal{A}^* \mapsto \mathcal{A}^*$ be an involutive antimorphism and $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be a Θ -standard word with seed. Then there exist an Arnoux-Rauzy word $\mathbf{v} \in \mathcal{B}^{\mathbb{N}}$ and a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$ such that*

$$\mathbf{u} = \varphi(\mathbf{v}) \text{ and } \#\mathcal{B} \leq \#\mathcal{A}.$$

All three mentioned theorems present almost Θ_1 -rich word as an image of a Θ_2 -rich word by a suitable morphism. The opposite question when a morphic image of a Θ_1 -rich word is almost Θ_2 -rich is not tackled here. In [12], a type of morphisms preserving the set of almost Θ_0 -rich words is studied.

2 Properties of words with finite Θ -defect

We will consider mainly infinite words $\mathbf{u} = (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ having their language $\mathcal{L}(\mathbf{u})$ closed under a given involutive antimorphism Θ . In other words, for any factor $w \in \mathcal{L}(\mathbf{u})$ we have $\Theta(w) \in \mathcal{L}(\mathbf{u})$.

For any factor $w \in \mathcal{L}(\mathbf{u})$ there exists an index i such that w is a prefix of the infinite word $u_i u_{i+1} u_{i+2} \dots$. Such an index is called an *occurrence* of w in \mathbf{u} . If each factor of \mathbf{u} has infinitely many occurrences in \mathbf{u} , the infinite word \mathbf{u} is said to be *recurrent*. It is easy to see that if the language of \mathbf{u} is closed under Θ , then \mathbf{u} is recurrent. For a recurrent infinite word \mathbf{u} , we may define the notion of a *complete return word* of any $w \in \mathcal{L}(\mathbf{u})$. It is a factor $v \in \mathcal{L}(\mathbf{u})$ such that w is a prefix and a suffix of v and w occurs in v exactly twice. Under a *return word* of a factor w we usually mean a word $q \in \mathcal{L}(\mathbf{u})$ such that qw is a complete return

word of w . If any factor $w \in \mathcal{L}(\mathbf{u})$ has only finitely many return words, then the infinite word \mathbf{u} is called *uniformly recurrent*.

An important role for the description of languages closed under Θ is played by the so-called super reduced Rauzy graphs $G_n(\mathbf{u})$. Before defining them, we will introduce some necessary notions.

We say that a factor $w \in \mathcal{L}(\mathbf{u})$ is left special (LS) if w has at least two left extensions, i.e., if there exist two letters $a, b \in \mathcal{A}$, $a \neq b$, such that $aw, bw \in \mathcal{L}(\mathbf{u})$. A right special (RS) factor is defined analogously. If a factor is LS and RS, we refer to it as bispecial. The closedness under Θ assures the following relation: a factor w is LS if and only if the factor $\Theta(w)$ is RS.

An n -simple path e is a factor of \mathbf{u} of length at least $n + 1$ such that the only special (right or left) factors of length n occurring in e are its prefix and suffix of length n . If w is the prefix of e of length n and v is the suffix of e of length n , we say that the n -simple path e begins with w and ends with v . We will denote by $G_n(\mathbf{u})$ an undirected graph whose set of vertices is formed by unordered pairs $(w, \Theta(w))$ such that $w \in \mathcal{L}(\mathbf{u})$, $|w| = n$, is RS or LS. We connect two vertices $(w, \Theta(w))$ and $(v, \Theta(v))$ by an unordered pair $(e, \Theta(e))$ if e or $\Theta(e)$ is an n -simple path beginning with w or $\Theta(w)$ and ending with v or $\Theta(v)$. Note that the graph $G_n(\mathbf{u})$ may have multiple edges and loops.

Surprisingly, the super reduced Rauzy graph $G_n(\mathbf{u})$ can be used to detect the equality in (2). Let us cite Corollary 7 from [13].

Proposition 4. *Let $n \in \mathbb{N}$ and $\mathcal{L}(\mathbf{u})$ be closed under Θ . Then $T_\Theta(n) = 0$ if and only if*

1. *all n -simple paths forming a loop in $G_n(\mathbf{u})$ are Θ -palindromes and*
2. *$G_n(\mathbf{u})$ after removing loops is a tree.*

Analogously to the case of the reversal mapping, one can see from the definition of Θ -defect that an infinite word \mathbf{u} has finite Θ -defect if and only if there exists an integer H such that of every prefix p of \mathbf{u} of length greater than H has a unioccurrent Θ -palindromic suffix, i.e., a suffix occurring exactly once in p . We will use this fact to prove the following lemma.

Lemma 5. *Let \mathbf{u} be a recurrent infinite word with finite Θ -defect. Then $\mathcal{L}(\mathbf{u})$ is closed under Θ .*

Proof. Let H be an integer such that every prefix of \mathbf{u} of length greater than H has a unioccurrent Θ -palindromic suffix. Suppose that w is a factor of \mathbf{u} such that $\Theta(w) \notin \mathcal{L}(\mathbf{u})$. Since \mathbf{u} is recurrent, we can find two consecutive occurrences i and j of the factor w such that $i, j > H$ and $i < j$. Denote p the prefix of \mathbf{u} ending with w occurring at j , i.e., $|p| = j + |w|$. Since $|p| > H$, there exists a unioccurrent Θ -palindromic suffix of p . Denote s to be such a suffix. If $|s| \leq |w|$, then s is a factor of w and thus occurs at least twice in p - a contradiction with the unioccurrence of s . If $|s| > |w|$, the w is a factor of s which is a Θ -palindrome and thus contains $\Theta(w)$ as well - a contradiction with the assumption that $\Theta(w) \notin \mathcal{L}(\mathbf{u})$.

In [3], various properties are shown for words with finite Θ_0 -palindromic defect. These properties and their proofs are valid even if we replace the antimorphism Θ_0 by an arbitrary Θ . Therefore, we mention here the relevant statements without proving them.

Proposition 6. *Let \mathbf{u} be an infinite recurrent word such that $D_\Theta(\mathbf{u}) < +\infty$. Then there exists a positive integer H such that*

- every prefix of \mathbf{u} longer than H has a unioccurrent Θ -palindromic suffix;
- for any factor $w \in \mathcal{L}(\mathbf{u})$ such that $|w| > H$, occurrences of w and $\Theta(w)$ in the word \mathbf{u} alternate;
- for any $w \in \mathcal{L}(\mathbf{u})$ such that $|w| > H$, every factor $v \in \mathcal{L}(\mathbf{u})$ beginning with w , ending with $\Theta(w)$, and with no other occurrences of w or $\Theta(w)$ is a Θ -palindrome;
- $T_\Theta(n) = 0$ for any integer $n > H$.

As already mentioned, the first property listed in the previous proposition, in fact, characterizes words with finite Θ -defect. We do not know whether this is the case of the remaining properties. If we restrict our attention to uniformly recurrent words, only then several characterizations of words with finite Θ -defect can be shown. The next proposition states two of them that we will use in what follows. Again, the proposition is based on the work done in [3] for $\Theta = \Theta_0$. No modifications besides replacing Θ_0 by Θ in its proof are needed, therefore, we will omit it.

Proposition 7. *Let \mathbf{u} be a uniformly recurrent infinite word with language closed under Θ . The following statements are equivalent.*

- $D_\Theta(\mathbf{u}) < +\infty$;
- there exists a positive integer K such that for any Θ -palindrome $w \in \mathcal{L}(\mathbf{u})$ of length $|w| \geq K$, all complete return words of w are Θ -palindromes;
- there exists a positive integer H such that for any $w \in \mathcal{L}(\mathbf{u})$, the longest Θ -palindromic suffix of w is unioccurrent in w .

A Θ -standard word with seed is an infinite word defined by using Θ -palindromic closure, for details see [8]. Construction of such word \mathbf{u} guarantees that \mathbf{u} is uniformly recurrent (cf. Proposition 3.5. in [8]). The authors of [8] showed (Proposition 4.8) that any complete return word of a sufficiently long Θ -palindromic factor is a Θ -palindrome as well. Therefore, Θ -standard words with seed serve as an example of almost Θ -rich words.

Corollary 8. *Let \mathbf{u} be a Θ -standard word with seed. Then $D_\Theta(\mathbf{u}) < +\infty$.*

3 Proofs

In this section we give proofs of all three theorems stated in Introduction. Although Theorem 2 seems to be only a refinement of Theorem 1, constructions of the morphisms φ in their proofs differ substantially. It is caused by stronger properties we may exploit for a uniformly recurrent word.

Proof (Proof of Theorem 1). Recall that according to Lemma 5 the language $\mathcal{L}(\mathbf{u})$ is closed under Θ_1 .

If \mathbf{u} is an eventually periodic word with language closed under Θ_1 , then \mathbf{u} is purely periodic. Any purely periodic word is a morphic image of a word \mathbf{v} over one-letter alphabet under the morphism which assigns to this letter the period of \mathbf{u} . Therefore we may assume without loss of generality that \mathbf{u} is not eventually periodic.

Since $D_{\Theta_1}(\mathbf{u}) < +\infty$, according to Propositions 4 and 6, there exists $H \in \mathbb{N}$ such that

1. $\forall w \in \mathcal{L}(\mathbf{u}), |w| > H$, occurrences of w and $\Theta_1(w)$ alternate;
2. $\forall w \in \mathcal{L}(\mathbf{u}), |w| > H$, every factor beginning with w , ending with $\Theta_1(w)$ and with no other occurrences of w or $\Theta_1(w)$ is a Θ_1 -palindrome;
3. $\forall n \geq H$, every loop in $G_n(\mathbf{u})$ is a Θ_1 -palindrome and $G_n(\mathbf{u})$ after removing loops is a tree.

Fix $n > H$. If an edge $(b, \Theta_1(b))$ in $G_n(\mathbf{u})$ is a loop, then, according to the property 3, we have $b = \Theta_1(b)$. If the edge $(b, \Theta_1(b))$ connects two distinct vertices $(w_1, \Theta_1(w_1))$ and $(w_2, \Theta_1(w_2))$, then there exist exactly two n -simple paths b and $\Theta_1(b)$ such that WLOG the n -simple path b begins with w_1 and ends with w_2 and the simple path $\Theta_1(b)$ begins with $\Theta_1(w_2)$ and ends with $\Theta_1(w_1)$.

We assign to every n -simple path b a new symbol $[b]$, i.e., we define the alphabet \mathcal{B} as

$$\mathcal{B} := \{[b] \mid b \in \mathcal{L}(\mathbf{u}) \text{ is an } n\text{-simple path}\}$$

and on this alphabet we define an involutive antimorphism $\Theta_2 : \mathcal{B}^* \mapsto \mathcal{B}^*$ in the following way:

$$\Theta_2([b]) := [\Theta_1(b)].$$

We are now going to construct a suitable infinite word $\mathbf{v} \in \mathcal{B}^{\mathbb{N}}$. Let $(s_i)_{i \in \mathbb{N}}$ denote a strictly increasing sequence of indices such that s_i is an occurrence of RS or LS factor of length n and every RS and LS factor of length n occurs at some index s_i . We define $\mathbf{v} = (v_i)_{i \in \mathbb{N}}$ by the formula

$$v_i = [b] \quad \text{if} \quad b = u_{s_i} u_{s_i+1} \dots u_{s_{i+1}+n-1}.$$

This construction can be done for any $n > H$. Since infinitely many prefixes of \mathbf{u} are LS or RS factors, we can choose such $n > H$ that the prefix of \mathbf{u} of length n is LS or RS, i.e., $s_0 = 0$.

According to Proposition 12 in [13], to prove that \mathbf{v} is Θ_2 -rich we need to show the following:

- (i) for every non-empty factor $w \in \mathcal{L}(\mathbf{v})$, any factor v beginning with w and ending with $\Theta_2(w)$, with no other occurrences of w or $\Theta_2(w)$, is a Θ_2 -palindrome;
- (ii) for every letter $[b] \in \mathcal{B}$ such that $[b] \neq \Theta_2([b])$, the occurrences of $[b]$ and $\Theta_2([b])$ in the word \mathbf{v} alternate.

Let us first verify (i). Let e and f be factors of \mathbf{v} such that e is a prefix and $\Theta_2(e)$ is a suffix of f and there are no other occurrences of e or $\Theta_2(e)$ in f . In that case there exist integers $r \leq k$ such that $f = [b_1][b_2] \dots [b_k]$ and $e = [b_1][b_2] \dots [b_r]$. The case $r = k$ is trivial. Suppose $r < k$. Since \mathbf{v} is defined as a coding of consecutive occurrences of n -simple paths in \mathbf{u} , factor f codes a certain segment of the word \mathbf{u} . Let us denote that segment $F = u_j \dots u_l$ where $j = s_t$ for some $t \in \mathbb{N}$ and $l = s_{t+k-1} + n - 1$. Factor e codes in the same way a factor $E = u_j \dots u_h$ where $h = s_{t+r-1} + n - 1$.

Due to the definition of Θ_2 , the fact that e is a prefix of f and $\Theta_2(e)$ is a suffix of f ensures that E is a prefix of F and $\Theta(E)$ is a suffix of F . Suppose f is not a Θ_2 -palindrome. This implies that F is not a Θ_1 -palindrome which contradicts the property 3.

Let us now verify (ii). Consider $[b] \in \mathcal{B}$ such that $[b] \neq \Theta_2([b])$. Moving along the infinite word $\mathbf{u} = u_0 u_1 u_2 \dots$ from the left to the right with a window of width n corresponds to a walk in the graph $G_n(\mathbf{u})$. The pair b and $\Theta_1(b)$ of n -simple paths in \mathbf{u} represents an edge in $G_n(\mathbf{u})$ connecting two distinct vertices. Moreover, moving along the n -simple paths b and moving along $\Theta_1(b)$ can be viewed as traversing that edge in opposite directions. Since $G_n(\mathbf{u})$ after removing loops is a tree, the only way to traverse an edge is alternately in one direction and in the other. Thus, the occurrences of letters $[b]$ and $\Theta_2([b])$ in \mathbf{v} alternate.

We have shown that \mathbf{v} is Θ_2 -rich. It is now obvious how to define a morphism $\varphi : \mathcal{B}^* \mapsto \mathcal{A}^*$. If an n -simple path b equals $b = u_{s_i} u_{s_i+1} \dots u_{s_{i+1}+n-1}$, then we set $\varphi([b]) := u_{s_i} u_{s_i+1} \dots u_{s_{i+1}-1}$.

Proof (Proof of Theorem 2). Recall again that according to Lemma 5 the language $\mathcal{L}(\mathbf{u})$ is closed under Θ .

Next, we show that infinitely many Θ -palindromes are also prefixes of \mathbf{u} . Consider an integer H whose existence is guaranteed by Proposition 6 and denote by w a prefix of \mathbf{u} longer than H . Since occurrences of factors w and $\Theta(w)$ in \mathbf{u} alternate, according to the same proposition, the prefix of \mathbf{u} ending with the first occurrence of $\Theta(w)$ is a Θ -palindrome.

Let us denote by p a Θ -palindromic prefix of \mathbf{u} with length $|p| > K$ where K is the constant from Proposition 7. All complete return words of p are Θ -palindromes. Since \mathbf{u} is uniformly recurrent, there exist only finite number of complete return words to p . Let $r^{(1)}, r^{(2)}, \dots, r^{(M)}$ be the list of all these return words. Any complete return word $r^{(i)}$ has the form $q^{(i)}p = r^{(i)}$ for some factor $q^{(i)}$, usually called return word of p . Since $r^{(i)}$ and p are Θ -palindromes, we have

$$p\Theta(q^{(i)}) = q^{(i)}p \quad \text{for any return word } q^{(i)}. \quad (3)$$

Let us define a new alphabet $\mathcal{B} = \{1, 2, \dots, M\}$ and morphism $\varphi : \mathcal{B}^* \rightarrow \mathcal{A}^*$ by the prescription

$$\varphi(i) = q^{(i)}, \quad \text{for } i = 1, 2, \dots, M.$$

First we will check the validity of the relation

$$\Theta(\varphi(w)p) = \varphi(\Theta(w))p \quad \text{for any } w \in \mathcal{B}^*. \quad (4)$$

Let $w = i_1 i_2 \dots i_n$. Then $\Theta(\varphi(i_1 i_2 \dots i_n)p)$ equals to

$$\Theta(p)\Theta(\varphi(i_n))\Theta(\varphi(i_{n-1}))\dots\Theta(\varphi(i_1)) = p\Theta(q^{(i_n)})\Theta(q^{(i_{n-1})})\dots\Theta(q^{(i_1)})$$

and we may apply gradually n times the equality (3) to rewrite the right-hand side as

$$q^{(i_n)}q^{(i_{n-1})}\dots q^{(i_1)}p = \varphi(i_n)\varphi(i_{n-1})\dots\varphi(i_1)p = \varphi(\Theta_0(i_1 i_2 \dots i_n))p.$$

This proves the relation (4).

An important property of the morphism φ is its injectivity. Indeed, in accordance with the definition, number of occurrences of the factor p in $\varphi(w)p$ equals to the number of letters in w plus one. Moreover, each occurrence of p in $\varphi(w)p$ indicates beginning of an image of a letter under φ . Therefore $\varphi(w)p = \varphi(v)p$ necessarily implies $w = v$.

Let us finally define the word \mathbf{v} . As p is a prefix of \mathbf{u} , the word \mathbf{u} can be written as a concatenation of return words $q^{(i)}$ and thus we can determine a sequence $\mathbf{v} = (v_n) \in \mathcal{B}^{\mathbb{N}}$ such that

$$\mathbf{u} = q^{(v_0)}q^{(v_1)}q^{(v_2)}\dots$$

Directly from the definition of \mathbf{v} we have $\mathbf{u} = \varphi(\mathbf{v})$. Since \mathbf{u} is uniformly recurrent, the word \mathbf{v} is uniformly recurrent as well. To prove that \mathbf{v} is a Θ_0 -rich word, we will show that any complete return word of any Θ_0 -palindrome in the word \mathbf{v} is a Θ_0 -palindrome as well. According to Theorem 2.14 in [12], this implies the Θ_0 -richness of \mathbf{v} .

Let s be a Θ_0 -palindrome in \mathbf{v} and w its complete return word. Then $\varphi(w)p$ has precisely two occurrences of the factor $\varphi(s)p$. Since s is a Θ_0 -palindrome, we have according to (4) that $\varphi(s)p$ is a Θ -palindrome of length $|\varphi(s)p| \geq |p| > K$. Therefore $\varphi(w)p$ is a complete return word of a long enough Θ -palindrome and according to our assumption $\varphi(w)p$ is a Θ -palindrome as well. Therefore by using (4) we have

$$\varphi(w)p = \Theta(\varphi(w)p) = \varphi(\Theta_0(w))p$$

and injectivity of φ gives $w = \Theta_0(w)$, as we claimed.

Theorem 6.1 in [6] states that every standard Θ -episturmian word is an image of a standard episturmian word. Again, the role of Θ_0 can be perceived as more important. Also, compared to Theorem 2, it may be seen as a special case since Θ -episturmian words, according to Corollary 8, have finite Θ -defect.

Proof (Proof of Theorem 3).

If \mathbf{u} is periodic, then the claim is trivial. Suppose \mathbf{u} is aperiodic.

We are going to repeat the proof of Theorem 2 with a more specific choice of p . Theorem 4.4 in [8] implies that there exists $L \in \mathbb{N}$ such that any LS factor of \mathbf{u} longer than L is a prefix of \mathbf{u} . Without loss of generality, we may assume that the constant L is already chosen in such a way that all prefixes of \mathbf{u} longer than L have the same left extensions. Let us denote their number by M . According

to the same theorem, infinitely many prefixes of \mathbf{u} are Θ -palindromes and thus bispecial factors as well.

According to Corollary 8, \mathbf{u} has finite Θ -palindromic defect. Let K be the constant from Proposition 7. Altogether, there exists a bispecial factor p , $|p| > \max\{L, K\}$, such that it is a prefix of \mathbf{u} and a Θ -palindrome. Since p is longer than K , all complete return words to p are Θ -palindromes. As p is the unique left special factor of length $|p|$ in \mathbf{u} , its return words (i.e., complete return words after erasing the suffix p) end with distinct letters. It means that there are exactly M return words of p , denoted again $q^{(i)}$. Let us recall that by M we denoted number of left extensions of some factor, therefore $M \leq \#\mathcal{A}$.

The construction of the word \mathbf{v} and prescription of the morphism φ over the alphabet $\mathcal{B} = \{1, 2, \dots, M\}$ can be done in exactly the same way as in the proof of Theorem 2. It remains to show that \mathbf{v} is an Arnoux-Rauzy word.

According to Theorem 2 we know that \mathbf{v} is Θ_0 -rich and uniformly recurrent. Applying Lemma 5 we deduce that the language $\mathcal{L}(\mathbf{v})$ is closed under reversal.

Suppose there exist $v, w \in \mathcal{L}(\mathbf{v})$, two LS factors such that $|v| = |w|$ and $v \neq w$. Since the words $q^{(i)}$ end with distinct letters, it is clear that $\varphi(w)p$ is a LS factor of \mathbf{u} and it has the same number of left extensions as w . The same holds for $\varphi(v)p$. Since both these factors have their length greater than or equal to $|p| > L$ and are both LS, one must be prefix of another. Let WLOG $\varphi(w)p$ be a prefix of $\varphi(v)p$, i.e., $\varphi(v)p = \varphi(wv')p$. The injectivity of φ implies $w' = \varepsilon$ and thus $v = w$ – a contradiction.

Remark 9. Theorem 3 can be seen as a generalization of Theorem 6.1 in [6] to Θ -standard words with seed.

Remark 10. Note also that the proof of Theorem 3 is in fact a combination of methods used in preceding proofs of Theorems 1 and 2 in the sense that the set of complete return words $r^{(i)}$ of the factor p and the set of $|p|$ -simple paths in \mathbf{u} coincide.

4 Acknowledgement

We acknowledge financial support by the Czech Science Foundation grant GAČR 201/09/0584, by the grants MSM6840770039 and LC06002 of the Ministry of Education, Youth, and Sports of the Czech Republic, and by the grant of the Grant Agency of the Czech Technical University in Prague.

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